Semi-Classical Quantum Fields Theories and Frobenius Manifolds

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Abstract

We show that a semi-classical quantum field theory comes with a versal family with the property that the corresponding partition function generates all path integrals and satisfies a system of 2nd order differential equations determined by algebras of classical observables. This versal family gives rise to a notion of special coordinates that is analogous to that in string theories. We also show that for a large class of semi-classical theories, their moduli space has the structure of a Frobenius super-manifold.

1 Introduction

Frobenius (formal super-)manifolds [4] were obtained by abstracting properties of Topological Conformal Field Theory in 2-dimensions [12, 3], but the mathematical structure was observed much earlier by K. Saito as certain flat structures on moduli spaces of unfolding of isolated singularities [10]. Saito's examples are physically relevant to topological Landau-Ginzburg models. Another large class of Frobenius manifolds has been constructed by Barannikov and Kontsevich [1] originally in the context of moduli space of topological string B model with Calabi-Yau target space [13]. (For a concise review of three constructions of Frobenius manifolds, see [7].)

The purpose of this paper is to study natural based moduli spaces of semi-classical Quantum Field Theories (QFTs) in general. The tangent space to the moduli space at the base point is the space of equivalence classes of observables of the given semi-classical QFT. We shall show that a large class of semi-classical QFTs has the structure of (formal) Frobenius super-manifold on their moduli space. This is the positive answer to a question posed in [8].

In the Batalin-Vilkovisky (BV) scheme of Quantum Field Theory (QFT) one constructs a BV-algebra $(\mathscr{C}, \Delta, \cdot)$ with an associated BV bracket $(\ ,\ )$, where $(\mathscr{C}, \cdot)$ is

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graded commutative algebra of functions on the space C of all fields and anti-fields and Δ is an odd second-order operator, whose failure to be a derivation of the product $\cdot$ is measured by the BV bracket [2, 11]. A semi-classical BV master action functional S is a ghost number zero element $S \in \mathscr{C}^0$, whose restriction to the space of classical fields is the usual classical action functional, and satisfies the so-called semi-classical BV master equation:

$$(S, S) = 0, \qquad \Delta S = 0.$$

Semi-classical BV master action functional S defines an operator $Q := (S,\ )$, which satisfies $Q^2 = 0$ and is a derivation of the product $\cdot$. The restriction of Q to the space of fields is the usual BRST operator. Then cohomology of the complex $(\mathscr{C}, Q)$ corresponds to equivalence classes of classical observables and is a graded commutative algebra called *classical algebra of observables*. A classical observable is a quantum observable if it belongs to Ker Δ. Throughout this paper we assume that the cohomology of the complex $(\mathscr{C}, Q)$ is finite dimensional for each degree (the ghost number).

It is natural to consider family of semi-classical QFTs defined by family of semi-classical action functionals $\mathscr{S} = S + \mathcal{O}$ satisfying the semi-classical master equation $\Delta\mathscr{S} = (\mathscr{S}, \mathscr{S}) = 0$. Equivalently

$$\begin{aligned} \Delta\mathcal{O} &= 0, \\ Q\mathcal{O} + \frac{1}{2}(\mathcal{O}, \mathcal{O}) &= 0. \end{aligned} \tag{1.1}$$

An infinitesimal solution of the above equation corresponds to a representative of Q-cohomology, which is annihilated by Δ, namely a classical observable which is also a quantum observable. The motivation for considering a family of semi-classical action functional is to define and study a generating functional of all path integrals of the given semi-classical field theory. The generating functional is the partition function of a versal semi-classical action functional $\mathscr{S} = S + \mathcal{O}$ whose infinitesimal terms span all quantum observables.

We shall call a QFT semi-classical if its master action functional satisfies the semi-classical BV master equation and if every equivalence class of classical observable has a representative belonging to Ker Δ. The later requirement in the above is natural since it is a necessary condition for a versal family of semi-classical BV master action functionals. A main result of this paper is that this is also a sufficient condition to have a versal family. We also construct a special versal master action functional $\mathscr{S} = S + \mathcal{O}$ for which the corresponding partition function $\mathfrak{Z}$ satisfies the following 2nd order differential equation:

$$-\hbar \frac{\partial^2 \mathfrak{Z}}{\partial t^\alpha \partial t^\beta} + \mathcal{A}_{\alpha\beta}{}^{\gamma} \frac{\partial \mathfrak{Z}}{\partial t^\gamma} = 0$$

determined by the structure constants $\{\mathcal{A}_{\alpha\beta}{}^{\gamma}\}$ of the classical algebras of observables parameterized by the moduli space. This can be viewed as an alternative to the usual approach of studying the quantum field theory, which compute path integrals via summing over Feynman diagrams after suitable gauge fixing procedure.

Furthermore, we show that the tangent space to moduli space of semi-classical QFTs has (i) a linear pencil of torsion free flat connection and (ii) a flat invariant metric, provided that semi-classical fields theory also comes with a BK-integral—an idea intro-

duced by Barannikov-Kontsevich [1]. Combining (i) and (ii), the moduli space of semi-classical QFTs with a BK-integral has the structure of a (formal) Frobenius manifold. This can be viewed as a generalization of the construction of Barannikov-Kontsevich, which applies to a smaller class of semi-classical QFTs associated with a differential BV-algebra $(\mathscr{C}, \Delta, Q, \cdot, (\ ,\ ))$ satisfying with ΔQ-property.

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2 Classical and Quantum Algebras

In this section we introduce the notion of a semi-classical quantum algebra, which is extracted from a semi-classical version of Batalin-Vilkovisky (BV) quantization scheme of field theory. We use the Einstein summation convention throughout. Fix a ground field $\Bbbk$ of characteristic zero. A $\mathbb{Z}$-graded super-commutative associative algebra over $\Bbbk$ is a pair $(\mathscr{C}, \cdot)$, where $\mathscr{C} = \bigoplus_{k\in\mathbb{Z}} \mathscr{C}^k$ is a $\mathbb{Z}$-graded $\Bbbk$-module and $\cdot$ is a super-commutative, associative product; we say a homogeneous element $a \in \mathscr{C}^k$ carries the ghost number k, and use $|a|$ to denote the ghost number of a. The ground field $\Bbbk$ is assigned to have the ghost number 0. A super-commutative and associative product $\cdot : \mathscr{C}^{k_1} \otimes \mathscr{C}^{k_2} \longrightarrow \mathscr{C}^{k_1+k_2}$ is a $\Bbbk$-bilinear map of ghost number 0 satisfying super-commutativity: $a \cdot b = (-1)^{|a||b|} b \cdot a$ for any homogeneous elements a and b, and associativity: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

DEFINITION 1 A differential Batalin-Vilkovisky (dBV) algebra over $\Bbbk$ is a quadruple $(\mathscr{C}, \cdot, \Delta, Q)$ where

1. $(\mathscr{C}, \cdot)$ is a $\mathbb{Z}$-graded super-commutative associative $\Bbbk$- algebra with unit,

2. $(\mathscr{C}, Q, \cdot)$ is a (super)-Commutative Differential Graded Algebra (CDGA) over k; $Q : \mathscr{C}^k \longrightarrow \mathscr{C}^{k+1}$ is a $\Bbbk$-linear operator of ghost number 1, satisfying $Q^2 = 0$ and is a derivation of the product, i.e.,
$$Q(a \cdot b) = (Qa) \cdot b + (-1)^{|a|} a \cdot (Qb),$$

3. $\Delta : \mathscr{C}^k \longrightarrow \mathscr{C}^{k+1}$ is a $\Bbbk$-linear operator of ghost number 1, satisfying $\Delta^2 = 0$ which is not a derivation of the product. The failure of Δ to be a derivation of $\cdot$ is measured by the so-called BV bracket $(\ ,\ ) : \mathscr{C}^{k_1} \otimes \mathscr{C}^{k_2} \longrightarrow \mathscr{C}^{k_1+k_2+1}$ defined by
$$(-1)^{|a|}(a, b) := \Delta(a \cdot b) - \Delta a \cdot b - (-1)^{|a|} a \cdot \Delta b,$$
which is a derivation of the product (super-Poisson law)
$$(a, b \cdot c) = (a, b) \cdot c + (-1)^{(|a|+1)|b|} b \cdot (a, c),$$

4. $\Delta Q + Q\Delta = 0$.

A BV algebra is a dBV algebra with the differential $Q = 0$.

PROPOSITION 2 *For a dBV algebra $(\mathscr{C}, \cdot, \Delta, Q)$ with associated BV bracket $(\ ,\ )$,*

1. *the triple $(\mathscr{C}, Q, (\ ,\ ))$ is a differential graded Lie algebra (DGLA) over $\Bbbk$; i.e., $Q : \mathscr{C}^k \longrightarrow \mathscr{C}^{k+1}$ and $Q^2 = 0$ (both by the Definition 1 above) such that Q is a derivation the bracket*

$$Q(a, b) = (Qa, b) + (-1)^{|a|+1}(a, Qb),$$

and satisfies the following super-commutativity

$$(a, b) = -(-1)^{(|a|+1)(|b|+1)}(b, a),$$

and the super-Jacobi law

$$(a, (b, c)) = ((a, b), c) + (-1)^{(|a|+1)(|b|+1)}(b, (a, c)).$$

2. *Δ is a derivation of the BV bracket;*

$$\Delta(a, b) = (\Delta a, b) + (-1)^{|a|+1}(a, \Delta b).$$

Proof of the above proposition is standard.

DEFINITION 3 For a dBV algebra $(\mathscr{C}, \cdot, \Delta, Q)$ over $\Bbbk$ with associated BV bracket $(\ ,\ )$, we call the quadruple $\big(\mathscr{C}, \cdot, Q, (\ ,\ )\big)$ a classical algebra and we call the complex $(\mathscr{C}, Q)$ a classical complex.

The dBV-algebra first appeared in the context of the notion of a *semi-classical* BV master action functional as follows Consider a BV-algebra $(\mathscr{C}, \Delta, \cdot)$ with associated BV bracket $(\ ,\ )$. A *semi-classical* BV master action functional is a ghost number zero element $S \in \mathscr{C}^0$,which satisfies the semi-classical BV master equation $(S, S) = \Delta S = 0$. Defining $Q = (S,\ )$ it follows that $\Delta Q + Q\Delta = Q^2 = 0$, so that the quadruple $(\mathscr{C}, \cdot, \Delta, Q)$ is a dBV algebra with associated BV bracket $(\ ,\ )$.

Now we are ready to define the notion of a quantum algebra in the BV quantization scheme

DEFINITION 4 A quantum algebra in the BV quantization scheme (a BV quantum algebra) is a triple $(\mathscr{C}[[\hbar]], \boldsymbol{K}, \cdot)$ where

1. the pair $(\mathscr{C}[[\hbar]], \cdot)$ is a $\mathbb{Z}$-graded super-commutative associative $\Bbbk[[\hbar]]$-algebra with unit, $\hbar$ is assigned to carry the ghost number 0
2. the pair $(\mathscr{C}[[\hbar]], \boldsymbol{K})$ is a complex over $\Bbbk[[\hbar]]$, i.e., $\boldsymbol{K} : \mathscr{C}[[\hbar]]^k \longrightarrow \mathscr{C}[[\hbar]]^{k+1}$ and $\boldsymbol{K}^2 = 0$,
3. the operator $\boldsymbol{K}$ is not a derivation of the product $\cdot$ but the failure of $\boldsymbol{K}$ from being a derivation of the product is a derivation divisible by $\hbar$ which is a Lie bracket, i.e., for any homogeneous elements $\boldsymbol{a}, \boldsymbol{b} \in \mathscr{C}[[\hbar]]$ we have

$$\boldsymbol{K}(\boldsymbol{a} \cdot \boldsymbol{b}) - (\boldsymbol{K}\boldsymbol{a}) \cdot \boldsymbol{b} - (-1)^{|\boldsymbol{a}|}\boldsymbol{a} \cdot (\boldsymbol{K}\boldsymbol{a}) := -\hbar(-1)^{|\boldsymbol{a}|}(\boldsymbol{a}, \boldsymbol{b})$$

for some Lie bracket $(\ ,\ ) : \mathscr{C}[[\hbar]]^{k_1} \otimes \mathscr{C}[[\hbar]]^{k_2} \longrightarrow \mathscr{C}[[\hbar]]^{k_1+k_2+1}$ satisfying $(\boldsymbol{a}, \boldsymbol{b} \cdot \boldsymbol{c}) = (\boldsymbol{a}, \boldsymbol{b}) \cdot \boldsymbol{c} + (-1)^{(|\boldsymbol{a}+1)|\boldsymbol{b}||}\boldsymbol{b} \cdot (\boldsymbol{a}, \boldsymbol{c})$.

4. The bracket $(\ ,\ )$ itself does not depend on $\hbar$, that is $(\mathscr{C}, \mathscr{C}) \subset \mathscr{C}$.

We call the complex $(\mathscr{C}[[\hbar]], \boldsymbol{K})$ quantum complex.

It follows that the triple $(\mathscr{C}[[\hbar]], \boldsymbol{K}, (\ ,\ ))$ is a DGLA over $\Bbbk[[\hbar]]$, which can be proved as Proposition 2. By quantum master equation we means

$$-\hbar \boldsymbol{K} e^{-\mho/\hbar} = 0 \Longleftrightarrow \boldsymbol{K}\mho + \frac{1}{2}(\mho, \mho) = 0$$

for $\mho \in \mathscr{C}[[\hbar]]^0$, where we used the identity $-\hbar \boldsymbol{K} e^{-\mho/\hbar} = e^{\mho/\hbar}\big(\boldsymbol{K}\mho + \frac{1}{2}(\mho, \mho)\big)$. It also follows that the restriction $Q = \boldsymbol{K}\big|_{\mathscr{C}}$ of $\boldsymbol{K}$ on $\mathscr{C}$ is a derivation of the product.

COROLLARY 5 *Let $(\mathscr{C}[[\hbar]], \boldsymbol{K}, \cdot)$ be a quantum algebra (over $\Bbbk[[\hbar]]$) with the associated bracket $(\ ,\ )$ and let $Q = \boldsymbol{K}\big|_{\mathscr{C}}$, then the quadruple $(\mathscr{C}, Q, \cdot, (\ ,\ ))$ is a classical algebra.*

PROPOSITION 6 *Let $(\mathscr{C}, \cdot, \Delta, Q)$ be a dBV algebra over $\Bbbk$ with associated BV bracket $(\ ,\ )$. Then the triple $(\mathscr{C}[[\hbar]], \boldsymbol{K} := -\hbar\Delta + Q, \cdot)$ is a quantum algebra.*

Proof. The condition $\Delta^2 = \Delta Q + Q\Delta = Q^2 = 0$ implies that $\boldsymbol{K}^2 = 0$. $\boldsymbol{K}$ with respect to the product satisfies the relation in Def. 4.2., as Q and Δ do. We have

$$\boldsymbol{K}(\boldsymbol{a} \cdot \boldsymbol{b}) - \boldsymbol{K}\boldsymbol{a} \cdot \boldsymbol{b} - (-1)^{|\boldsymbol{a}|}\boldsymbol{a} \cdot (\boldsymbol{K}\boldsymbol{b}) = -\hbar(-1)^{|\boldsymbol{a}|}(\boldsymbol{a}, \boldsymbol{b})$$

since Q is a derivation of the product., etc.. □

A dBV algebra $(\mathscr{C}, \cdot, \Delta, Q)$ with associated BV bracket $(\ ,\ )$ gives both a classical algebra $(\mathscr{C}, Q, \cdot, (\ ,\ ))$ and a quantum algebra $(\mathscr{C}[[\hbar]], \boldsymbol{K} = -\hbar\Delta + Q, \cdot)$. We call the former the classical limit of the latter. Cohomology of the classical complex $(\mathscr{C}, Q)$ corresponds to equivalence classes of classical observables, while the cohomology of quantum complex $(\mathscr{C}[[\hbar]], \boldsymbol{K} = -\hbar\Delta + Q)$ corresponds to equivalence classes of quantum observables.

DEFINITION 7 A dBV algebra $(\mathscr{C}, \Delta, \cdot, Q)$ is called a semi-classical algebra if every cohomology class of the complex $(\mathscr{C}, Q)$ has a representative belonging to Ker Δ.

For a quantum algebra formed from a semi-classical differential BV algebra, every classical cohomology class has a representative which is also a quantum observable.

EXAMPLE 8 We say a dBV-algebra $(\mathscr{C}, \Delta, Q, \cdot)$ has the ΔQ-property if

$$(\text{Ker } Q \cap \text{Ker } \Delta) \cap (\text{Im } \Delta \oplus \text{Im } Q) = \text{Im } \Delta Q = \text{Im } Q\Delta.$$

That is, for any $x \in \mathscr{C}$ satisfying $Qx = 0$ and $x = \Delta y$ for some $y \in \mathscr{C}$, then there exist some $z \in \mathscr{C}$ such that $x = Q\Delta z = -\Delta Q z$ etc... Such a dBV-algebra is also semi-classical, i.e., for any Q-cohomology class $\big[y\big]$ there is a representative y' such that $\Delta y' = 0$. To see this, consider any representative y of the Q-cohomology class $[y]$ such that $\Delta y = x \neq 0$; we have $Qy = 0$ by definition, which leads to $\Delta Q y = 0$. Using $\Delta Q = -Q\Delta$, we have $Q\Delta y = Qx = 0$. Thus $Qx = 0$ and $x = \Delta y$ and, from the assumption there is some z such that $x = \Delta y = \Delta Q z$. It follows that $\Delta(y - Qz) = 0$ so for $y' := y - Qz$, $\big[y'\big] = \big[y\big]$ and $\Delta y' = 0$.

Note, however, that a semi-classical algebra *does not necessarily* satisfy the ΔQ-property. Here is an innocent example:

EXAMPLE 9 Let $\mathscr{C} = \Bbbk[x^1,\ldots,x^m,\eta_1,\ldots,\eta_m]$ be a super-polynomial algebra with free associative product subject to the super-commutative relations $x^i\cdot x^j = x^j\cdot x^i$, $x^i\cdot\eta_j = \eta_j\cdot x^i$ and $\eta_i\cdot\eta_j = -\eta_j\cdot\eta_i$ Assign ghost number 0 to $\{x^i\}$ and -1 to $\{\eta_i\}$. Then $\mathscr{C} = \mathscr{C}^{-m}\oplus\cdots\oplus\mathscr{C}^{-1}\oplus\mathscr{C}^0$. Note that $\mathscr{C}^0 = \Bbbk[x^1,\ldots,x^m]$. Define $\Delta := \frac{\partial^2}{\partial x^i\eta_i} : \mathscr{C}^\bullet \longrightarrow \mathscr{C}^{\bullet+1}$. It is obvious that $\Delta^2 = 0$, and the triple $(\mathscr{C},\Delta,\cdot)$ is a BV algebra over $\Bbbk$. We also note that $\mathscr{C}^0 \in \text{Im }\Delta$. To see this, it suffices to consider an arbitrary monomial $(x^1)^{N_1}\cdots(x^m)^{N_m} \in \mathscr{C}^0 = \Bbbk[x^1,\ldots,x^m]$ and observe that, for instance,

$$(x^1)^{N_1}\cdots(x^m)^{N_m} = \frac{1}{(N_1+1)}\Delta\left(\eta_1\cdot(x^1)^{N_1+1}\cdots(x^m)^{N_m}\right).$$

For any $S\in\mathscr{C}^0$, we always have $\Delta S = (S,S) = 0$. Fix S and define $Q = (S,\ )$, then the quadruple $(\mathscr{C},\Delta,Q,\cdot)$ is a dBV algebra. Denote H the cohomology of the complex $(\mathscr{C},Q)$. It is easy to determine that

$$H^0 = \Bbbk[x^1,\ldots,x^m]\Big/\left\langle\frac{\partial S}{\partial x^1},\cdots,\frac{\partial S}{\partial x^m}\right\rangle,$$

since $\mathscr{C}^0 \subset \text{Ker }Q$, and any element $R\in\mathscr{C}^{-1}$ with the ghost number -1 is in the form $R = R^i\eta_i$, where $\{R_i\}$ is a set of m elements in $\mathscr{C}^0$, such that $QR = R^i\frac{\partial S}{\partial x^i}$. Now we assume that S is a polynomial (in $x's$) with isolated singularities, so that the cohomology H of the complex $(\mathscr{C},Q)$ is concentrated in the ghost number zero part, i.e., $H = H^0$. Then any representative of H belongs to Ker Δ, since $\mathscr{C}^0$ itself belongs to Ker Δ. Thus the dBV algebra is obviously semi-classical. We already know that $\mathscr{C}^0 \subset \text{Ker }Q\cap\text{Im}\Delta$, so that ΔQ-property would implies that $\mathscr{C}^0 \subset \text{Im }Q\Delta$ and, in particular, $H^0 = 0$, which is not generally true.

3 Special Coordinates on the Moduli Space of Semi-Classical QFTs

Consider a semi-classical algebra $(\mathscr{C},\Delta,Q,\cdot)$ with associated BV bracket $(\ ,\ )$. Let $\{O_\alpha\}$ be a set of representative of a basis of the (total) cohomology H of the complex $(\mathscr{C},Q)$ satisfying $\Delta O_\alpha = 0$ for all $\forall\alpha$. Let $\{t^\alpha\}$, where $t^\alpha\in H^*$ be the dual basis such that $gh(t^\alpha)+gh(O_\alpha) = 0$. Denote by $\Bbbk[[H^*]]$ the algebra of formal power series on the $\mathbb{Z}$-graded vector space H^*, the linear dual of H. Then,

PROPOSITION 10 *There exist an solution $\mathcal{O}$ to the equation*

$$Q\mathcal{O}+\frac{1}{2}(\mathcal{O},\mathcal{O}) = 0, \tag{3.1}$$

in formal power series with values in $\mathscr{C}$

$$\mathcal{O} = t^\alpha O_\alpha + \sum_{n=2}^{\infty}\frac{1}{n!}t^{\alpha_n}\cdots t^{\alpha_1}\mathcal{O}_{\alpha_1\cdots\alpha_n} \in (\mathscr{C}\otimes\Bbbk[[H^*]])^0$$

satisfying

1. **Versality**: $\{O_\alpha\}$ *forms a basis of the cohomology of the complex* $(\mathscr{C}, Q)$
2. **Speciality**: *for* $\mathcal{O}_\alpha := \frac{\partial}{\partial t^\alpha}\mathcal{O}$ *and* $\mathcal{O}_{\alpha\beta} := \frac{\partial^2}{\partial t^\alpha \partial t^\beta}\mathcal{O}$, $\mathcal{O}_\alpha \in$ *Ker* Δ,

$$\begin{aligned} \mathcal{O}_{\alpha\beta} &= \Delta\Lambda_{\alpha\beta} \\ \mathcal{O}_\alpha \cdot \mathcal{O}_\beta &= \mathcal{A}_{\alpha\beta}{}^\gamma \mathcal{O}_\gamma + Q\Lambda_{\alpha\beta} + (\mathcal{O}, \Lambda_{\alpha\beta}). \end{aligned} \tag{3.2}$$

We call such $\mathcal{O}$ *a special versal solution.*

Note that if $\mathcal{O}$ is a special versal solution to equation (3.1) then

$$\Delta\mathcal{O} = 0. \tag{3.3}$$

Our proof is constructive and we placed it in Section 4 for a better presentation. Here we examine some of consequences of the above proposition. We shall explain the terminology speciality in Section 3.2.

3.1 Linear Pencil of Torsion Free Flat Connections

Begin with an obvious consequence.

COROLLARY 11 *If* $(\mathscr{C}, \Delta, Q, \cdot, (\ ,\ ))$ *is a dBV algebra for which each cohomology class of the complex* $(\mathscr{C}, Q)$ *has a representative in Ker* Δ *then the differential graded Lie algebra* $(\mathscr{C}, Q, (\ ,\ ))$ *is formal and the associated extended deformation functor (see [BK] for this notion) is representable by the algebra* $\mathbb{k}[[H^*]]$. *Hence, one has a smooth-formal graded moduli space.*

Let $\mathcal{Q} := Q + (\mathcal{O},\ ) : (\mathscr{C} \otimes \mathbb{k}[[H^*]])^k \longrightarrow (\mathscr{C} \otimes \mathbb{k}[[H^*]])^{k+1}$, where $\mathcal{O}$ is the versal solution. Then we have $\mathcal{Q}^2 = \mathcal{Q}\Delta + \Delta\mathcal{Q} = 0$ and then the quadruple $(\mathscr{C} \otimes \mathbb{k}[[H^*]], \cdot, \Delta, \mathcal{Q})$ which is also a semi-classical algebra called a versal semi-classical algebra. Equation (3.1) together with super-Jacobi identity implies that $\mathcal{Q}^2 = 0$, the super-Poisson law implies that $\mathcal{Q}$ is a derivation of the product, and the condition (3.3) implies that $\mathcal{Q}\Delta + \Delta\mathcal{Q} = 0$. Also apply $\frac{\partial}{\partial t^\alpha}$ to equation (3.1) to get $\Delta\mathcal{O}_\alpha = \mathcal{Q}\mathcal{O}_\alpha = 0$. Thus $\{\mathcal{O}_\alpha\}$ is a set of representatives of a basis of the total cohomology of the complex $(\mathscr{C} \otimes \mathbb{k}[[H^*]], \mathcal{Q})$ belonging to Ker Δ. The second relation in equation (3.2) now reads

$$\mathcal{O}_\alpha \cdot \mathcal{O}_\beta = \mathcal{A}_{\alpha\beta}{}^\gamma \mathcal{O}_\gamma + \mathcal{Q}\Lambda_{\alpha\beta}, \tag{3.4}$$

thus $\{\mathcal{A}_{\alpha\beta}{}^\gamma\}$ are the structure constants (in formal power series in $\{t^\rho\}$) of the algebra of the cohomology of the versal classical complex $(\mathscr{C} \otimes \mathbb{k}[[H^*]], \mathcal{Q})$. It also follows that we have corresponding versal quantum algebra $(\mathscr{C}[[\hbar]] \otimes \mathbb{k}[[\hbar]][[H^*]], \boldsymbol{\mathcal{K}}, \cdot)$, where $\boldsymbol{\mathcal{K}} := -\hbar\Delta + \mathcal{Q} = \boldsymbol{K} + (\mathcal{O},\ )$ satisfying $\boldsymbol{\mathcal{K}}^2 = 0$.

Now we state one of our main theorems, which is a corollary of Proposition 10. For simplicity, we use the convention that $(-1)^{|\alpha|}$ stand for $(-1)^{|O_\alpha|}$.

THEOREM 12 *The set of structure constants* $\{\mathcal{A}_{\alpha\beta}{}^\gamma\}$ *in formal power series in* $\{t^\rho\}$ *satisfies*

$$\begin{aligned} \mathcal{A}_{\alpha\beta}{}^\gamma - (-1)^{|\alpha||\beta|}\mathcal{A}_{\beta\alpha}{}^\gamma &= 0, \\ \mathcal{A}_{\beta\gamma}{}^\rho \mathcal{A}_{\alpha\rho}{}^\sigma - (-1)^{|\alpha||\beta|}\mathcal{A}_{\alpha\gamma}{}^\rho \mathcal{A}_{\beta\rho}{}^\sigma &= 0, \\ \partial_\alpha \mathcal{A}_{\beta\gamma}{}^\rho - (-1)^{|\alpha||\beta|}\partial_\beta \mathcal{A}_{\alpha\gamma}{}^\rho &= 0. \end{aligned}$$

In other words, $(\mathcal{A})_\beta{}^\gamma := dt^\alpha \mathcal{A}_{\alpha\beta}{}^\gamma$ can be viewed as a connection 1-form for a linear pencil of torsion-free flat connections.

Proof. The first relation in the theorem is a trivial consequence of the super-commutativity of the product $\cdot$; $\mathcal{O}_\alpha \cdot \mathcal{O}_\beta = (-1)^{|\alpha||\beta|}\mathcal{O}_\beta \cdot \mathcal{O}_\alpha$. To prove the second relation, consider the identity $\mathcal{O}_\alpha \cdot (\mathcal{O}_\beta \cdot \mathcal{O}_\gamma) - (-1)^{|\alpha||\beta|}\mathcal{O}_\beta \cdot (\mathcal{O}_\alpha \cdot \mathcal{O}_\gamma) = 0$ obtained by combining the associativity and super-commutativity of the product. By applying the relation (3.4), and the fact that $\mathcal{Q}$ is a derivation of the product, we obtain

$$\left(\mathcal{A}_{\beta\gamma}{}^\rho \mathcal{A}_{\alpha\rho}{}^\sigma - (-1)^{|\alpha||\beta|}\mathcal{A}_{\alpha\gamma}{}^\rho \mathcal{A}_{\beta\rho}{}^\sigma\right)\mathcal{O}_\sigma = \mathcal{Q}\mathcal{M}_{[\alpha\beta]\gamma}, \tag{3.5}$$

where

$$\begin{aligned}\mathcal{M}_{[\alpha\beta]\gamma} :=& -\mathcal{A}_{\beta\gamma}{}^\rho \Lambda_{\alpha\rho} + (-1)^{|\alpha||\beta|}\mathcal{A}_{\alpha\gamma}{}^\rho \Lambda_{\beta\rho} \\ & - (-1)^{|\alpha|}\mathcal{O}_\alpha \cdot \Lambda_{\beta\gamma} + (-1)^{|\alpha||\beta|+|\beta|}\mathcal{O}_\beta \cdot \Lambda_{\alpha\gamma}.\end{aligned}$$

Thus the linear combination $\left(\mathcal{A}_{\beta\gamma}{}^\rho \mathcal{A}_{\alpha\rho}{}^\sigma - (-1)^{|\alpha||\beta|}\mathcal{A}_{\alpha\gamma}{}^\rho \mathcal{A}_{\beta\rho}{}^\sigma\right)\mathcal{O}_\sigma$ vanishes in the $\mathcal{Q}$-cohomology. Furthermore every coefficient in the above linear combination must vanish, since the $\mathcal{Q}$-cohomology classes of $\{\mathcal{O}_\sigma\}$ form a basis of the total cohomology of the complex $(\mathscr{C} \otimes \Bbbk[[H^*]], \mathcal{Q})$. Thus we proved the second relation in the theorem and conclude, in turn, that $\mathcal{Q}\mathcal{M}_{[\alpha\beta]\gamma} = 0$ due to equation (3.5).

To prove the third relation in the theorem consider equation (3.4) in the form

$$\mathcal{O}_\beta \cdot \mathcal{O}_\gamma = \mathcal{A}_{\beta\gamma}{}^\rho \mathcal{O}_\rho + \mathcal{Q}\Lambda_{\beta\gamma}$$

and apply $\frac{\partial}{\partial t^\alpha}$ to each side and rearrange to get

$$\mathcal{C}_{\alpha\beta\gamma} = \partial_\alpha \mathcal{A}_{\beta\gamma}{}^\rho \mathcal{O}_\rho + (-1)^{|\alpha|}\mathcal{Q}\left(\partial_\alpha \Lambda_{\beta\gamma}\right), \tag{3.6}$$

where

$$\mathcal{C}_{\alpha\beta\gamma} := \mathcal{O}_{\alpha\beta} \cdot \mathcal{O}_\gamma + (-1)^{|\alpha||\beta|}\mathcal{O}_\beta \cdot \mathcal{O}_{\alpha\gamma} - \mathcal{A}_{\beta\gamma}{}^\rho \mathcal{O}_{\alpha\rho} - (\mathcal{O}_\alpha, \Lambda_{\beta\gamma}).$$

Let $\mathcal{C}_{[\alpha\beta]\gamma} := \mathcal{C}_{\alpha\beta\gamma} - (-1)^{|\alpha||\beta|}\mathcal{C}_{\beta\alpha\gamma}$, which explicit expression is

$$\begin{aligned}\mathcal{C}_{[\alpha\beta]\gamma} :=& -\mathcal{A}_{\beta\gamma}{}^\rho \mathcal{O}_{\alpha\rho} - (\mathcal{O}_\alpha, \Lambda_{\beta\gamma}) + (-1)^{|\alpha||\beta|}\mathcal{O}_\beta \cdot \mathcal{O}_{\alpha\gamma} \\ & + (-1)^{|\alpha||\beta|}\mathcal{A}_{\alpha\gamma}{}^\rho \mathcal{O}_{\beta\rho} + (-1)^{|\alpha||\beta|}\left(\mathcal{O}_\beta, \Lambda_{\alpha\gamma}\right) - \mathcal{O}_\alpha \cdot \mathcal{O}_{\beta\gamma}.\end{aligned}$$

Then equation (3.6) implies that

$$\begin{aligned}\mathcal{C}_{[\alpha\beta]\gamma} =& \left(\partial_\alpha \mathcal{A}_{\beta\gamma}{}^\rho - (-1)^{|\alpha||\beta|}\partial_\beta \mathcal{A}_{\alpha\gamma}{}^\rho\right)\mathcal{O}_\rho \\ & + (-1)^{|\alpha|}\mathcal{Q}\left(\partial_\alpha \Lambda_{\beta\gamma} - (-1)^{|\alpha||\beta|}\partial_\beta \Lambda_{\alpha\gamma}\right).\end{aligned}$$

Note, the right hand side of the equation above is a $\Bbbk$-linear combination of $\{\mathcal{O}_\rho\}$ plus $\mathcal{Q}$-exact term. Observe that the third relation in the theorem would follow from looking at these coefficients if $\mathcal{C}_{[\alpha\beta]\gamma} \in \operatorname{Im}\mathcal{Q}$.

Using $\mathcal{O}_{\alpha\beta} = \Delta\Lambda_{\alpha\beta}$, the first relation in equation (3.2) for the *speciality* in Proposition 10, one can check that $\Delta\mathcal{M}_{[\alpha\beta]\gamma} = -\mathcal{C}_{[\alpha\beta]\gamma}$ after a direct computation. From

the condition that $\mathcal{Q}\mathcal{M}_{[\alpha\beta]\gamma} = 0$, we can express $\mathcal{M}_{[\alpha\beta]\gamma}$ as a $\Bbbk$-linear combination of $\{\mathcal{O}_\rho\}$ plus a $\mathcal{Q}$-exact term:

$$\mathcal{M}_{[\alpha\beta]\gamma} = \mathcal{A}_{[\alpha\beta]\gamma}{}^{\rho}\mathcal{O}_\rho + \mathcal{Q}\Lambda_{[\alpha\beta]\gamma},$$

for some structure constants $\{\mathcal{A}_{[\alpha\beta]\gamma}{}^{\rho}\}$ and some $\Lambda_{[\alpha\beta]\gamma}$. Apply Δ to each side above to obtain

$$\Delta\mathcal{M}_{[\alpha\beta]\gamma} = \mathcal{A}_{[\alpha\beta]\gamma}{}^{\rho}\Delta\mathcal{O}_\rho + \Delta\mathcal{Q}\Lambda_{[\alpha\beta]\gamma} = -\mathcal{Q}\Delta\Lambda_{[\alpha\beta]\gamma},$$

where we used $\Delta\mathcal{O}_\rho = 0$ and $\mathcal{Q}\Delta + \Delta\mathcal{Q} = 0$. Combining the above with the identity $\Delta\mathcal{M}_{[\alpha\beta]\gamma} = -\mathcal{C}_{[\alpha\beta]\gamma}$ we conclude that $\mathcal{C}_{[\alpha\beta]\gamma} = \mathcal{Q}\Delta\Lambda_{[\alpha\beta]\gamma}$. Thus $\mathcal{C}_{[\alpha\beta]\gamma} \in \text{Im } \mathcal{Q}$ and the third relation in the theorem follows. □

3.2 Generating Functional for Path Integrals

Consider the semi-classical algebra $(\mathscr{C}, \cdot, \Delta, Q)$ as with a special versal solution $\mathcal{O}$ as in Proposition 10 . Then $\mathcal{O}$ is also an versal solution to the quantum master equation of the quantum algebra $(\mathscr{C}[[\hbar]], \boldsymbol{K} = -\hbar\Delta + Q, \cdot)$, i.e,

$$-\hbar\boldsymbol{K}e^{-\mathcal{O}/\hbar} = 0 \Longleftrightarrow \boldsymbol{K}\mathcal{O} + \frac{1}{2}(\mathcal{O},\mathcal{O}) = 0.$$

which is equivalent to $\Delta\mathcal{O} = Q\mathcal{O} + \frac{1}{2}(\mathcal{O},\mathcal{O}) = 0$ as $\mathcal{O}$ does not depend on $\hbar$. Applying $\partial_\beta := \frac{\partial}{\partial t^\beta}$ to the above identity we have

$$\boldsymbol{K}\left(\mathcal{O}_\beta \cdot e^{-\mathcal{O}/\hbar}\right) = 0 \Longrightarrow \boldsymbol{K}\mathcal{O}_\beta + (\mathcal{O},\mathcal{O}_\beta) = 0,$$

which is equivalent to the conditions $\Delta\mathcal{O}_\beta = \mathcal{Q}\mathcal{O}_\beta = 0$. So $\{\mathcal{O}_\beta\}$ is a set of representative of a basis of the total cohomology of the complex $(\mathscr{C}[[\hbar]] \otimes \Bbbk[[\hbar]][[H^*]], \boldsymbol{\mathcal{K}})$. We note that the two conditions in equation (3.2) of Proposition 10 imply the following relation

$$\mathcal{O}_\alpha \cdot \mathcal{O}_\beta - \hbar\mathcal{O}_{\alpha\beta} = \mathcal{A}_{\alpha\beta}{}^{\gamma}\mathcal{O}_\gamma + \boldsymbol{\mathcal{K}}\Lambda_{\alpha\beta}, \tag{3.7}$$

where $\boldsymbol{\mathcal{K}}\Lambda_{\alpha\beta} = -\hbar\Delta\Lambda_{\alpha\beta} + \mathcal{Q}\Lambda_{\alpha\beta}$, since, for instance, $\Delta\Lambda_{\alpha\beta} = \mathcal{O}_{\alpha\beta}$. We also note that the above relation may also be rewritten in the following form

$$\hbar^2\frac{\partial^2}{\partial t^\alpha \partial t^\beta}\, e^{-\mathcal{O}/\hbar} = -\hbar\mathcal{A}_{\alpha\beta}{}^{\gamma}\frac{\partial}{\partial t^\gamma}e^{-\mathcal{O}/\hbar} + \boldsymbol{K}\left(\Lambda_{\alpha\beta} \cdot e^{-\mathcal{O}/\hbar}\right). \tag{3.8}$$

Now we illustrate what is special about a special versal solution. Let $\mathcal{O}$ be an versal solution to equation (3.1), i.e., $Q\mathcal{O} + \frac{1}{2}(\mathcal{O},\mathcal{O}) = 0$, but without the speciality condition. Then $\mathcal{O}$ may not, in general, satisfies $\Delta\mathcal{O} = 0$. Assume that $\mathcal{O}$ does satisfy $\Delta\mathcal{O} = 0$ as well, so that $-\hbar\boldsymbol{K}e^{-\mathcal{O}/\hbar} = 0$. Then $\mathcal{O}_\alpha \in \text{Ker } \Delta$ and $Q\mathcal{O}_\alpha + (\mathcal{O},\mathcal{O}_\alpha) = 0$, so that $\boldsymbol{\mathcal{K}}\mathcal{O}_\alpha = 0$, where $\boldsymbol{\mathcal{K}} = \boldsymbol{K} + (\mathcal{O},\ )$ satisfying $\boldsymbol{\mathcal{K}}^2 = 0$. So $\{\mathcal{O}_\beta\}$ is a set of representative of a basis of the total cohomology of the complex $(\mathscr{C}[[\hbar]] \otimes \Bbbk[[\hbar]][[H^*]], \boldsymbol{\mathcal{K}})$. From $\hbar^2\partial_\alpha\partial_\beta\boldsymbol{K}e^{-\mathcal{O}/\hbar} = 0$, we deduce that $\boldsymbol{\mathcal{K}}\left(\mathcal{O}_\alpha \cdot \mathcal{O}_\beta - \hbar\mathcal{O}_{\alpha\beta}\right) = 0$. Thus

$$\mathcal{O}_\alpha \cdot \mathcal{O}_\beta - \hbar\mathcal{O}_{\alpha\beta} = \boldsymbol{\mathcal{A}}_{\alpha\beta}{}^{\gamma}\mathcal{O}_\gamma + \boldsymbol{\mathcal{K}}\boldsymbol{\Lambda}_{\alpha\beta},$$

for some formal power series $\boldsymbol{\mathcal{A}}_{\alpha\beta}{}^{\gamma} = \mathcal{A}^{(0)\gamma}_{\alpha\beta} + \hbar\mathcal{A}^{(1)\gamma}_{\alpha\beta} + \cdots$ in $\Bbbk[[\hbar]][[H^*]]$ and for some $\boldsymbol{\Lambda}_{\alpha\beta} = \Lambda^{(0)}_{\alpha\beta} + \hbar\Lambda^{(1)}_{\alpha\beta} + \cdots \in \left(\mathscr{C}[[\hbar]] \otimes \Bbbk[[\hbar]][[H^*]]\right)^{|\mathcal{O}_\alpha|+|\mathcal{O}_\beta|-1}$. Then the

above relation implies that (in the classical limit) $\mathcal{O}_\alpha \cdot \mathcal{O}_\beta = \mathcal{A}_{\alpha\beta}^{(0)\gamma}\mathcal{O}_\gamma + \mathcal{Q}\Lambda_{\alpha\beta}^{(0)}$ so that the formal power series $\mathcal{A}_{\alpha\beta}^{(0)\gamma}$ are the structure constants the super-commutative associative $\Bbbk[[H^*]]$-algebra structure on the cohomology of the versal classical complex $(\mathscr{C} \otimes \Bbbk[[H^*]], \mathcal{Q})$. Finally, if $\mathcal{O}$ is a special versal solution as well, then $\mathcal{O}_\alpha \in \text{Ker}\,\Delta$ and $\mathcal{O}_{\alpha\beta} = \Delta\Lambda_{\alpha\beta}{}^{(0)}$, and we have $\boldsymbol{\mathcal{A}}_{\alpha\beta}{}^\gamma = \mathcal{A}_{\alpha\beta}^{(0)\gamma}$.

In the BV quantization scheme the quantum master equation is interpreted as the condition that path integral does not depends of a choice of gauge fixing. More precisely, a gauge fixing corresponds to a choice of a Lagrangian subspace in the space of all "fields" and "anti-fields". The path integral is supposed to be performed over such a Lagrangian subspace and the quantum BV master equation is supposed to be the condition that such path integral does not depends on a smooth change of the Lagrangian subspace.

Alternatively we can view the path integral of a Quantum Field Theory as a linear map from $\boldsymbol{K}$ cohomology of associated quantum algebra $(\mathscr{C}[[\hbar]], \boldsymbol{K}, \cdot)$ to the ground ring; this is especially concrete when the $\boldsymbol{K}$-cohomology is finite dimensional for each ghost numbers, which is an assumption that we make. In the presence of a special versal solution $\mathcal{O}$ to the (semi-classical) quantum master equation and the associated versal quantum algebra $(\mathscr{C}[[\hbar]] \otimes \Bbbk[[\hbar]][[H^*]], \boldsymbol{\mathcal{K}} := \boldsymbol{K} + (\mathcal{O},\ ), \cdot)$, we may consider the left cyclic module $\boldsymbol{\mathcal{N}} := (\mathscr{C}[[\hbar]] \otimes \Bbbk[[\hbar]][[H^*]]) \cdot e^{-\mathcal{O}/\hbar}$ generated by $e^{-\mathcal{O}/\hbar}$ and define an versal Feynman path integral as a $\Bbbk[[\hbar]][[H^*]]$ module map, which we denote by

$$\oint\!\!\!\!\!\!\mathscr{P}\, \boldsymbol{X} \cdot e^{-\mathcal{O}/\hbar}$$

for any $\boldsymbol{X} \in (\mathscr{C}[[\hbar]] \otimes \Bbbk[[\hbar]][[H^*]])$ with the property that

$$\oint\!\!\!\!\!\!\mathscr{P}\, \boldsymbol{K}\left(\boldsymbol{Y} \cdot e^{-\mathcal{O}/\hbar}\right) \equiv \oint\!\!\!\!\!\!\mathscr{P}\, (\boldsymbol{\mathcal{K}}\boldsymbol{Y}) \cdot e^{-\mathcal{O}/\hbar} = 0 \tag{3.9}$$

for any $\boldsymbol{Y} \in (\mathscr{C}[[\hbar]] \otimes \Bbbk[[\hbar]][[H^*]])$. We, then, define corresponding partition function $\boldsymbol{\mathcal{Z}}$ by

$$\boldsymbol{\mathcal{Z}} := \oint\!\!\!\!\!\!\mathscr{P}\, e^{-\mathcal{O}/\hbar}$$

From equation (3.8), we have

$$\left(\hbar^2 \frac{\partial^2}{\partial t^\alpha \partial t^\beta} + \hbar \mathcal{A}_{\alpha\beta}{}^\gamma \frac{\partial}{\partial t^\gamma}\right) \oint\!\!\!\!\!\!\mathscr{P}\, e^{-\mathcal{O}/\hbar} = \oint\!\!\!\!\!\!\mathscr{P}\, \boldsymbol{K}\left(\Lambda_{\alpha\beta} \cdot e^{-\mathcal{O}/\hbar}\right)$$

and from property (3.9) and the definition of $\boldsymbol{\mathcal{Z}}$, we have

$$\left(\hbar^2 \frac{\partial^2}{\partial t^\alpha \partial t^\beta} + \hbar \mathcal{A}_{\alpha\beta}{}^\gamma \frac{\partial}{\partial t^\gamma}\right) \boldsymbol{\mathcal{Z}} = 0. \tag{3.10}$$

3.3 The Flat Metric on the Moduli Space

In this subsection we show that the formal moduli space associated with a semi-classical dBV algebra naturally carries the structure of a formal Frobenius super-manifold provided the dBV algebra is semi-classical and comes with an additional structure called

BK integral. This result is a generalization of the main result in [1], which applies to the smaller class of semi-classical dBV algebras having the ΔQ-property. We shall follow some relevant parts in the papers [1, 7] closely. Also we do not attempt to explain formal Frobenius super-manifolds, but refer the reader to [1].

DEFINITION 13 Let $(\mathscr{C}, \Delta, Q, \cdot)$ be a differential BV algebra over $\mathbb{k}$ with associated BV bracket $(\ ,\ )$. An even $\mathbb{k}$-linear functional $\int : \mathscr{C} \longrightarrow \mathbb{k}$ is called BK integral if it satisfies the following two properties;

(i) $\forall a, b \in \mathscr{C}$ we have $\int (Qa) \cdot b = -(-1)^{|a|} \int a \cdot (Qb)$,

(ii) $\forall a, b \in \mathscr{C}$ we have $\int (\Delta a) \cdot b = (-1)^{|a|} \int a \cdot (\Delta b)$.

COROLLARY 14 *Let $(\mathscr{C}, \Delta, Q, \cdot)$ be a semi-classical dBV algebra over $\mathbb{k}$ with a BK integral $\int$. Assume that $\mathcal{O} \in (\mathscr{C} \otimes \mathbb{k}[[H^*]])^0$ is an versal solution of semi-classical master equation $\Delta\mathcal{O} = Q\mathcal{O} + \frac{1}{2}(\mathcal{O}, \mathcal{O}) = 0$, then $\int$ is also an BK integral of an versal semi-classical algebra $(\mathscr{C} \otimes \mathbb{k}[[H^*]], \Delta, \mathcal{Q} = Q + (\mathcal{O},\ ), \cdot)$ as an even $\mathbb{k}[[H^*]]$-linear functional $\int : \mathscr{C} \otimes \mathbb{k}[[H^*]] \longrightarrow \mathbb{k}[[H^*]]$.*

Proof. It is suffice to show that $\int \mathcal{Q}a = 0$ for $\forall a \in \mathscr{C} \otimes S(H^*)$;

$$\int \mathcal{Q}a = \int Qa + \int (\mathcal{O}, a) = \int (\mathcal{O}, a) = \int \Delta(\mathcal{O} \cdot a) - \int \mathcal{O} \cdot \Delta a = -\int (\Delta\mathcal{O}) \cdot a = 0,$$

where we used $\int Qb = \int \Delta b = 0$ for any b by (i) and (ii) after setting $a = 1$ and $\Delta\mathcal{O} = 0$ by assumption. Note also that $\mathcal{Q}$ is a derivation of the product $\cdot$, implying that $0 = \int \mathcal{Q}(a \cdot b) = \int (\mathcal{Q}a) \cdot b + (-1)^{|a|} \int a \cdot (\mathcal{Q}b)$. $\square$

Consider a semi-classical dBV algebra with a BK integral $\int$. Let $\mathcal{O}$ be a special versal solution as in Proposition 10, so that $\mathcal{O}_\alpha := \partial_\alpha \mathcal{O} \in \text{Ker}\,\Delta$ and $\mathcal{O}_{\alpha\beta} = \partial_\alpha \mathcal{O}_\beta = \Delta\Lambda_{\alpha\beta}$.

CLAIM 15 *Denote $g_{\alpha\beta} := \int \mathcal{O}_\alpha \cdot \mathcal{O}_b$ and $\mathcal{A}_{\alpha\beta\gamma} := \mathcal{A}_{\alpha\beta}{}^\rho g_{\rho\gamma}$. Then $\partial_\gamma g_{\alpha\beta} = 0$ and $\mathcal{A}_{\alpha\beta\gamma} = (-1)^{|\alpha|(|\beta|+|\gamma|)} \mathcal{A}_{\beta\gamma\alpha}$.*

Proof. Applying $\partial_\gamma := \frac{\partial}{\partial t^\gamma}$ to the definition of $g_{\alpha\beta}$, yields

$$\begin{aligned}
\partial_\gamma g_{\alpha\beta} &= \int (\partial_\gamma \mathcal{O}_\alpha) \cdot \mathcal{O}_b + (-1)^{|\alpha||\gamma|} \int \mathcal{O}_\alpha \cdot (\partial_\gamma \mathcal{O}_b) \\
&= \int (\Delta\Lambda_{\gamma\alpha}) \cdot \mathcal{O}_\beta + (-1)^{|\alpha||\gamma|} \int \mathcal{O}_\alpha \cdot (\Delta\Lambda_{\gamma\beta}) \\
&= 0,
\end{aligned}$$

since $\Delta\mathcal{O}_\alpha = 0$ for all α and $\int (\text{Im}\,\Delta) \cdot (\text{Ker}\,\Delta) = 0$. For the second claim we first note that

$$\int (\mathcal{O}_\alpha \cdot \mathcal{O}_\beta) \cdot \mathcal{O}_\gamma = \int (\mathcal{A}_{\alpha\beta}{}^\rho \mathcal{O}_\rho + \mathcal{Q}\Lambda_{\alpha\beta}) \cdot \mathcal{O}_\gamma = \mathcal{A}_{\alpha\beta}{}^\rho \int \mathcal{O}_\rho \cdot \mathcal{O}_\gamma = \mathcal{A}_{\alpha\beta}{}^\rho g_{\rho\gamma} = \mathcal{A}_{\alpha\beta\gamma},$$

where we used $\mathcal{Q}\mathcal{O}_\gamma = 0$ and the fact that $\mathcal{Q}$ is a derivation of the product. Similarly,

$$(-1)^{|\alpha|(|\beta|+|\gamma|)} \int (\mathcal{O}_\beta \cdot \mathcal{O}_\gamma) \cdot \mathcal{O}_\alpha = (-1)^{|\alpha|(|\beta|+|\gamma|)} \mathcal{A}_{\beta\gamma\alpha}.$$

So the second claim is equivalent to

$$\int (\mathcal{O}_\alpha \cdot \mathcal{O}_\beta) \cdot \mathcal{O}_\gamma = (-1)^{|\alpha|(|\beta|+|\gamma|)} \int (\mathcal{O}_\beta \cdot \mathcal{O}_\gamma) \cdot \mathcal{O}_\alpha,$$

which is obvious due the super-commutativity and associativity of the product. $\square$

Combining the identities in Theorem 12 and the above Claim, we have

THEOREM 16 *The moduli space of a semi-classical QFT equipped with a BK integral carries a structure of formal Frobenius super-manifold.*

It seems natural to expect that there be a mirror phenomenon of semi-classical QFT with BK integral involved in the general study of morphisms of formal Frobenius structures.

4 Proof of Proposition 10

We begin by explaining our plan for proof. Instead of solving the semi-classical master equation (3.1) directly we concurrently build a pair of inductive systems

$$\begin{array}{ccccccccccccc} I_0 & \subset & I_1 & \subset & I_2 & \subset & \cdots & \subset & I_{n-1} & \subset & I_n & \subset & \cdots \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \cdots & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \cdots \\ J_0 & \subset & J_1 & \subset & J_2 & \subset & \cdots & \subset & J_{n-1} & \subset & J_n & \subset & \cdots \end{array}$$

The inductive systems consist of

- $I_0 = \left(\left\{\mathcal{O}^{[0]}_\alpha\right\}\right)$: fix a set $\{O_\alpha\}$ of representative of a homogeneous basis of the cohomology of the complex $(\mathscr{C}, Q)$ satisfying $\Delta O_\alpha = 0$ and define $\mathcal{O}^{[0]}_\alpha := O_\alpha$.
- $J_{n-1} = \left(\left\{\Lambda^{[0]}_{\alpha\beta}, \mathcal{A}^{[0]}_{\alpha\beta}{}^\gamma\right\}, \left\{\Lambda^{[1]}_{\alpha\beta}, \mathcal{A}^{[1]}_{\alpha\beta}{}^\gamma\right\} \ldots, \left\{\Lambda^{[n-1]}_{\alpha\beta}, \mathcal{A}^{[n-1]}_{\alpha\beta}{}^\gamma\right\}\right)$, where $\mathcal{A}^{[k]\gamma}_{\alpha\beta}$ is the part of the algebra structure constant of word-length k in $\{t^\rho\}$ and $\Lambda^{[k]}_{\alpha\beta} \in \left(\mathscr{C} \otimes S^k(H^*)\right)^{|O_\alpha|+|O_\beta|-1}$, which together satisfy the following system of equations

$$Q\Lambda^{[\ell]}_{\alpha\beta} = \sum_{j=0}^{\ell} \mathcal{O}^{[j]}_\alpha \cdot \mathcal{O}^{[\ell-j]}_\beta - \sum_{j=1}^{\ell-1} \left(\mathcal{O}^{[\ell-j]}, \Lambda^{[j]}_{\alpha\beta}\right) - \sum_{j=0}^{\ell} \mathcal{A}^{[\ell-j]\gamma}_{\alpha\beta} \mathcal{O}^{[j]}_\gamma, \quad (4.1)$$

  for all $\ell = 0, 1, \ldots, n-1$, where $\mathcal{O}^{[\ell+1]}_\alpha := \frac{1}{(\ell+1)} t^\beta \Delta \Lambda^{[\ell]}_{\beta\alpha}$.
- $I_n = \left(\left\{\mathcal{O}^{[0]}_\alpha\right\}, \left\{\mathcal{O}^{[1]}_\alpha\right\}, \cdots, \left\{\mathcal{O}^{[n]}_\alpha\right\}\right)$, where $\mathcal{O}^{[k]}_\alpha \in \left(\mathscr{C} \otimes S^k(H^*)\right)^{|O_\alpha|}$, which satisfy the following set of equations

$$Q\mathcal{O}^{[k]}_\alpha = -\sum_{j=1}^{k} \left(\mathcal{O}^{[j]}, \mathcal{O}^{[k-j]}_\alpha\right), \quad (4.2)$$

  for all $k = 0, 1, \ldots, n$, where $\mathcal{O}^{[k+1]} := \frac{1}{(k+1)} t^\alpha \mathcal{O}^{[k]}_\alpha$

The following elementary lemma says that building I_n is equivalent to solving the Maurer-Cartan (MC) equation $Q\mathcal{O} + \frac{1}{2}(\mathcal{O}, \mathcal{O}) = 0$ modulo t^{n+2}.

LEMMA 17 *Let $\mathcal{O}^{[j+1]} := \frac{1}{(j+1)} t^\alpha \mathcal{O}^{[j]}_\alpha$ for $j = 0, 1, \ldots, n$. Let k be any integer with $0 \leq k \leq n$, then the conditions in (4.2) imply that*

$$Q\mathcal{O}^{[k+1]} + \frac{1}{2}\sum_{j=1}^{k}(\mathcal{O}^{[k-j+1]}, \mathcal{O}^{[j]}) = 0.$$

Proof. Multiply $(-1)^{|\alpha|}t^\alpha$ to the both hand-sides of the given condition and sum over α to get

$$\sum_{j=0}^{k-1}\left(\mathcal{O}^{[k-j]}, t^\alpha \mathcal{O}^{[j]}_\alpha\right) = -Qt^\alpha\mathcal{O}^{[k]}_\alpha,$$

which gives

$$\sum_{j=0}^{k-1}(j+1)\left(\mathcal{O}^{[k-j]}, \mathcal{O}^{[j+1]}\right) = -(k+1)Q\mathcal{O}^{[k+1]}.$$

The left hand side of the above equation can be rewritten as

$$\sum_{j=1}^{k} j(\mathcal{O}^{[k+1-j]}, \mathcal{O}^{[j]}) \equiv \frac{1}{2}(k+1)\sum_{j=1}^{k}(\mathcal{O}^{[k+1-j]}, \mathcal{O}^{[j]}),$$

using $(\mathcal{O}^{[k+1-j]}, \mathcal{O}^{[j]}) = (\mathcal{O}^{[j]}, \mathcal{O}^{[k+1-j]})$ and re-summing. □

We remark that the inductive system $I_0 \subset I_1 \subset \cdots \subset I_n \subset$ by itself is rather standard. Assume that one has built I_n, equivalently, that one has constructed a solution of the MC equation of a DGLA modulo t^{n+2}. The obstruction for the inclusion $I_n \subset I_{n+1}$ lies on the cohomology of the DGLA, and the obstruction does not depends on the choice of particular solution modulo t^{n+2}. If the obstruction vanishes, one can build an I_{n+1} by making certain choice. In our case the existence of a J_n shall be used to show that (i) the obstruction for an inclusion $I_n \subset I_{n+1}$ vanishes and (ii) there is a special choice for I_{n+1}. Specifically, I_{n+1} shall be obtained by J_n by applying Δ.

Now we are going to build the above mentioned pair of inductive systems.

- $I_0 \longrightarrow J_0$; Let $\mathcal{L}^{[0]}_{\alpha\beta} := \mathcal{O}^{[0]}_\alpha \cdot \mathcal{O}^{[0]}_\beta$, defined terms of I_0, which is Q-closed since Q is a derivation of the product. Thus we can express $\mathcal{L}^{[0]}_{\alpha\beta}$ as

  $$\mathcal{L}^{[0]}_{\alpha\beta} = \mathcal{A}^{[0]\gamma}_{\alpha\beta}\mathcal{O}^{[0]}_\gamma + Q\Lambda^{[0]}_{\alpha\beta}$$

  for some structure constants $\mathcal{A}^{[0]\gamma}_{\alpha\beta}$ and for some $\Lambda^{[0]}_{\alpha\beta} \in \mathscr{C}^{|O_\alpha|+|O_\beta|-1}$, which is defined modulo Ker Q. Fix a set $\{\Lambda^{[0]}_{\alpha\beta}\}$ once and for all. Then we have $J_0 = \left(\left\{\mathcal{A}^{[0]}_{\alpha\beta}{}^\gamma, \Lambda^{[0]}_{\alpha\beta}\right\}\right)$ satisfying

  $$\mathcal{O}^{[0]}_\alpha \cdot \mathcal{O}^{[0]}_\beta - \mathcal{A}^{[0]}_{\alpha\beta}{}^\gamma\mathcal{O}^{[0]}_\gamma = Q\Lambda^{[0]}_{\alpha\beta}, \tag{4.3}$$

  which is the $n = 0$ case of (4.1).

- $J_0 \to I_1$; Apply Δ to the both hand-sides equation (4.3) and $\Delta O_\alpha = \Delta Q + Q\Delta = 0$ to get $(-1)^{|\alpha|}\left(\mathcal{O}^{[0]}_\alpha, \mathcal{O}^{[0]}_\beta\right) = -Q\Delta\Lambda^{[0]}_{\alpha\beta}$. After multiplying $(-1)^{|\alpha|}t^\alpha$ to the both hand-sides of the above equation and summing over α we get

$$\left(t^\alpha\mathcal{O}^{[0]}_\alpha, \mathcal{O}^{[0]}_\beta\right) = -Q\left(t^\alpha\Delta\Lambda^{[0]}_{\alpha\beta}\right),$$

which leads to $(\mathcal{O}^{[1]}, \mathcal{O}^{[0]}_\alpha) = -Q\left(t^\alpha\Delta\Lambda^{[0]}_{\alpha\beta}\right)$, where $\mathcal{O}^{[1]} := t^\alpha\mathcal{O}^{[0]}_\alpha$. By setting $\mathcal{O}^{[1]}_\beta = t^\alpha\Delta\Lambda^{[0]}_{\alpha\beta}$ we build $I_1 = \left(\left\{\mathcal{O}^{[0]}_\alpha\right\}, \left\{\mathcal{O}^{[1]}_\alpha\right\}\right)$ which satisfy the following system of equations:

$$\begin{aligned} Q\mathcal{O}^{[0]}_\alpha &= 0, \\ Q\mathcal{O}^{[1]}_\alpha &= -(\mathcal{O}^{[1]}, \mathcal{O}^{[0]}_\alpha), \end{aligned} \tag{4.4}$$

which is the $n = 1$ case of (4.2). Define $\mathcal{O}^{[2]} = \frac{1}{2}t^\alpha\mathcal{O}^{[1]}_\alpha$. The inclusion $I_0 \subset I_1$ is obvious.

Fix $n > 0$ and assume that we have J_{n-1} and I_n as described before;

$$\begin{array}{ccccccccccc} I_0 & \subset & I_1 & \subset & I_2 & \subset & \cdots & \subset & I_{n-1} & \subset & I_n \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \cdots & \nearrow & \downarrow & \nearrow & \\ J_0 & \subset & J_1 & \subset & J_2 & \subset & \cdots & \subset & J_{n-1} & & \end{array}.$$

We shall establish

$$\begin{array}{ccccccccccccc} I_0 & \subset & I_1 & \subset & I_2 & \subset & \cdots & \subset & I_{n-1} & \subset & I_n & \subset & I_{n+1} \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \cdots & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \\ J_0 & \subset & J_1 & \subset & J_2 & \subset & \cdots & \subset & J_{n-1} & \subset & J_n & & \end{array}.$$

- $I_n \longrightarrow J_n$; Let $\mathcal{L}^{[n]}_{\alpha\beta}$ be given by

$$\mathcal{L}^{[n]}_{\alpha\beta} := \sum_{k=0}^{n}\mathcal{O}^{[k]}_\alpha\cdot\mathcal{O}^{[n-k]}_\beta - \sum_{k=0}^{n-1}\left(\mathcal{O}^{[n-k]}, \Lambda^{[k]}_{\alpha\beta}\right) - \sum_{k=1}^{n}\mathcal{A}^{[n-k]\gamma}_{\alpha\beta}\mathcal{O}^{[k]}_\gamma,$$

which is a combination of I_n and J_{n-1}. Now we show that $Q\mathcal{L}^{[n]} = 0$. Using the fact that Q is a derivation of both the product and the bracket, we have

$$\begin{aligned} Q\mathcal{L}^{[n]}_{\alpha\beta} :=& \sum_{j=0}^{n}Q\mathcal{O}^{[j]}_\alpha\cdot\mathcal{O}^{[n-j]}_\beta + (-1)^{|\alpha|}\sum_{j=0}^{n}\mathcal{O}^{[n-j]}_\alpha\cdot Q\mathcal{O}^{[j]}_\beta \\ &- \sum_{j=0}^{n-1}\left(Q\mathcal{O}^{[n-j]}, \Lambda^{[j]}_{\alpha\beta}\right) + \sum_{j=0}^{n-1}\left(\mathcal{O}^{[n-j]}, Q\Lambda^{[j]}_{\alpha\beta}\right) - \sum_{j=1}^{n}\mathcal{A}^{[n-j]\gamma}_{\alpha\beta}Q\mathcal{O}^{[k]}_\gamma. \end{aligned}$$

Now we use assumptions (4.2) and (4.1) to obtain

$$\begin{aligned} Q\mathcal{L}^{[n]}_\alpha =& \sum_{k=0}^{n-1}\sum_{j=0}^{k}\left(\left(\mathcal{O}^{[n-k]}, \mathcal{O}^{[j]}_\alpha\cdot\mathcal{O}^{[k-j]}_\beta\right) - \mathcal{O}^{[j]}_\alpha\cdot\left(\mathcal{O}^{[n-k]}, \mathcal{O}^{[k-j]}_\beta\right) - \left(\mathcal{O}^{[n-k]}, \mathcal{O}^{[j]}_\alpha\right)\cdot\mathcal{O}^{[k-j]}_\beta\right) \\ &- \sum_{k=1}^{n}\left(Q\mathcal{O}^{[k]} + \frac{1}{2}\sum_{j=1}^{k-1}\left(\mathcal{O}^{[j]}, \mathcal{O}^{[k-j]}\right), \Lambda^{[n-k]}_{\alpha\beta}\right), \end{aligned}$$

after some simple re-summations, an application of the super-Jacobi identity and one cancellation between two terms involving $\mathcal{A}^{[j]\gamma}_{\alpha\beta}$. The first line of the right-hand sides of the above vanishes due to the super-Poisson law, and the second line also vanishes due to the identity (4.2). Thus $Q\mathcal{L}^{[n]}_{\alpha\beta} = 0$.

Consequently we can express $\mathcal{L}^{[n]}_{\alpha\beta}$ in terms of the set of fixed representatives $\{\mathcal{O}^{[0]}_{\gamma}\}$ of the cohomology classes of the complex $(\mathscr{C}, Q)$ modulo Q-exact terms:

$$\mathcal{L}^{[n]}_{\alpha\beta} = \mathcal{A}^{[n]\gamma}_{\alpha\beta}\mathcal{O}^{[0]}_{\gamma} + Q\Lambda^{[n]}_{\alpha\beta},$$

where $\left\{\mathcal{A}^{[n]\gamma}_{\alpha\beta}\right\}$ is a set of structure constants with word-length n and $\Lambda^{[n]}_{\alpha\beta} \in (\mathscr{C}\otimes S^n(H^*))^{|O_\alpha|+|O_\beta|-1}$ is defined modulo Ker Q. We fix a set $\{\Lambda^{[n]}_{\alpha\beta}\}$ once and for all. Now we set

$$J_n := \left(J_{n-1}, \left\{\mathcal{A}^{[n]\gamma}_{\alpha\beta}, \Lambda^{[n]}_{\alpha\beta}\right\}\right)$$

which satisfy, for all $k = 0, 1, \ldots, n$

$$\sum_{j=0}^{k} \mathcal{O}^{[j]}_{\alpha}\cdot\mathcal{O}^{[k-j]}_{\beta} - \sum_{j=0}^{k-1}\left(\mathcal{O}^{[k-j]}, \Lambda^{[j]}_{\alpha\beta}\right) - \sum_{j=1}^{k}\mathcal{A}^{[k-j]\gamma}_{\alpha\beta}\mathcal{O}^{[j]}_{\gamma} = \mathcal{A}^{[k]\gamma}_{\alpha\beta}\mathcal{O}^{[0]}_{\gamma} + Q\Lambda^{[k]}_{\alpha\beta} \tag{4.5}$$

by definition. The inclusion $J_{n-1}\subset J_n$ is obvious.

- $J_n \to I_{n+1}$ amounts to setting $\mathcal{O}^{[n+1]}_{\alpha} = \frac{1}{(n+1)}t^\beta\Delta\Lambda^{[n]}_{\beta\alpha}$ such that

$$I_{n+1} := \left(I_n, \left\{\mathcal{O}^{[n+1]}_{\beta}\right\}\right) = \left(\left\{\mathcal{O}^{[0]}_{\beta}\right\}, \ldots, \left\{\mathcal{O}^{[n]}_{\beta}\right\}, \left\{\mathcal{O}^{[n+1]}_{\beta}\right\}\right).$$

Note that $\mathcal{O}^{[k+1]}_{\alpha} = \frac{1}{(k+1)}t^\beta\Delta\Lambda^{[k]}_{\beta\alpha}$ for all $k = 0, 1, \ldots, n-1, n$ by assumption. To prove above assertion we need to establish that the relations

$$\sum_{j=0}^{k}\left(\mathcal{O}^{[k-j+1]}, \mathcal{O}^{[j]}_{\beta}\right) = -Q\mathcal{O}^{[k+1]}_{\beta}, \tag{4.6}$$

for all $k = 0, 1, \ldots, n$ - in particular $Q\mathcal{O}^{[n+1]}_{\alpha} = -\sum_{j=0}^{n}\left(\mathcal{O}^{[n-j]}, \mathcal{O}^{[j]}_{\alpha}\right)$. The above assertion is a consequence of the identities (4.5) for J_n: we apply Δ to the both hand-sides of the identities (4.5) for J_n. We have, for all $k = 0, 1, \ldots, n$

$$(-1)^{|\alpha|}\sum_{j=0}^{k}\left(\mathcal{O}^{[j]}_{\alpha}, \mathcal{O}^{[k-j]}_{\beta}\right) + \sum_{j=0}^{k-1}\left(\mathcal{O}^{[k-j]}, \Delta\Lambda^{[j]}_{\alpha\beta}\right) = -Q\Delta\Lambda^{[k]}_{\alpha\beta}, \tag{4.7}$$

where we used that $\Delta\mathcal{O}^{[\ell]}_{\alpha} = \Delta\mathcal{O}^{[\ell+1]} = 0$ for all $\ell = 0, 1, \ldots, n$. Multiplying both sides by $(-1)^\alpha t^\alpha$ and sum over α to get

$$\sum_{j=0}^{k}\left(t^\alpha\mathcal{O}^{[j]}_{\alpha}, \mathcal{O}^{[k-j]}_{\beta}\right) + \sum_{j=0}^{k-1}\left(\mathcal{O}^{[k-j]}, t^\alpha\Delta\Lambda^{[j]}_{\alpha\beta}\right) = -Qt^\alpha\Delta\Lambda^{[k]}_{\alpha\beta},$$

which gives

$$\sum_{j=0}^{k}(j+1)\left(\mathcal{O}^{[j+1]},\mathcal{O}_\beta^{[k-j]}\right)+\sum_{j=0}^{k-1}(j+1)\left(\mathcal{O}^{[k-j]},\mathcal{O}_\beta^{[j]}\right)=-(k+1)Q\mathcal{O}_\beta^{[k+1]},$$

for all $k=0,1,\dots,n$. The left hand side of the above reduces to

$$(k+1)\sum_{j=0}^{k}\left(\mathcal{O}^{[k-j+1]},\mathcal{O}_\beta^{[j]}\right)$$

after a re-summation, which proves that the relations (4.6) are satisfied.

From the pair of inductive systems, it follows that every obstruction vanishes and we have a solution $\mathcal{O}=\sum_{k=1}^{\infty}\mathcal{O}^{[k]}\in\left(\mathscr{C}\otimes\Bbbk[[H^*]]\right)^0$ of the semi-classical master equation such that $\mathcal{O}^{[1]}=t^a\mathcal{O}_\alpha^{[0]}=t^\alpha O_\alpha$, where the cohomology classes of $\{O_\alpha\}$ from a basis of (total) cohomology of the complex $(\mathscr{C},Q)$ and

$$\mathcal{O}^{[k+1]}:=\frac{1}{(k+1)}t^\alpha\mathcal{O}_\alpha^{[k]},\qquad \mathcal{O}_\alpha^{[k+1]}:=\frac{1}{(k+1)}t^\beta\Delta\Lambda_{\beta\alpha}^{[k]},$$

for all $k=0,1,\dots,\infty$. Note the above conditions are equivalent to

$$\begin{cases}\quad\mathcal{O}_\alpha^{[k]}=\dfrac{\partial}{\partial t^\alpha}\mathcal{O}^{[k+1]},\\ \Delta\Lambda_{\alpha\beta}^{[k]}=\dfrac{\partial}{\partial t^\alpha}\mathcal{O}_\beta^{[k+1]}\equiv\dfrac{\partial^2}{\partial t^\alpha\partial t^\beta}\mathcal{O}^{[k+2]},\end{cases}$$

for all $k\geq 0$. Thus $\mathcal{O}_\alpha:=\frac{\partial}{\partial t^\alpha}\mathcal{O}=\sum_{k=0}^{\infty}\mathcal{O}_\alpha^{[k]}$ and the system of equations (4.2) limit is equivalent to the equation $Q\mathcal{O}_\alpha+(\mathcal{O},\mathcal{O}_\alpha)=0$. Furthermore, $\mathcal{O}_\alpha\in\mathrm{Ker}\,\Delta$ by construction. Let $\Lambda_{\alpha\beta}:=\sum_{k=0}^{\infty}\Lambda_{\alpha\beta}^{[k]}$ then we have $\mathcal{O}_{\alpha\beta}:=\frac{\partial^2}{\partial t^\alpha\partial t^\beta}\mathcal{O}=\Delta\Lambda_{\alpha\beta}$ and the systems of equations (4.1) is equivalent to

$$\mathcal{O}_\alpha\cdot\mathcal{O}_\beta=\mathcal{A}_{\alpha\beta}{}^\gamma\mathcal{O}_\gamma+Q\Lambda_{\alpha\beta}+(\mathcal{O},\Lambda_{\alpha\beta}).$$

Thus we have proved the proposition. □

5 Further Developments

The referee suggested to this author that Section 4 might be simplified by using homological perturbation theory (HPT). We note that the paper [5] adopts HPT to reexamine the result of Barannikov-Kontsevich [1]. This author, however, doesn't have any concrete ideas along the lines of this suggestion.

Consider a quantum algebra in Definition 4 and call it anomaly-free if every cohomology class of the associated classical complex has an extension (or $\hbar$-correction) to a cohomology class of the quantum complex. The class of semi-classical quantum algebras considered in this paper is a particular class of anomaly-free quantum algebras. A generalization of this work to a anomaly-free quantum algebra shall be studied in a sequel [9].

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